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Spectral properties of Jacobi matrices by asymptotic analysis ☆

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Abstract

We consider two classes of Jacobi matrix operators in l^2 with zero diagonals and with weights of the form $n^\alpha + c_n$ for $0<\alpha\leqslant 1$ or of the form $n^\alpha + c_n n^{\alpha-1}$ for $\alpha>1$, where $\{c_n\}$ is periodic. We study spectral properties of these operators (especially for even periods), and we find asymptotics of some of their generalized eigensolutions. This analysis is based on some discrete versions of the Levinson theorem, which are also proved in the paper and may be of independent interest.

Keywords: Jacobi matrix; Spectral analysis; Generalized eigenvectors; Transfer matrix; Asymptotic behaviour; The Levinson theorem, The Carleman condition; Absolutely continuous spectrum; Pure point spectrum

0. Introduction

It is well-known that for many types of operators their spectral properties are tightly related to some properties of the so-called "generalized eigenvectors" (i.e. solutions of an equation which is formally similar to the eigenequation, but these solutions need not belong to the space on which the operator acts). In the case of infinite Jacobi matrix operators the subordination theory of Khan and Pearson [14]

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is a significant example of the above relation. In these cases the generalized eigenvectors are solutions of second-order linear difference equations.

The aim of this paper is to find asymptotics of these "generalized eigensolutions", and to show some interesting spectral consequences of the asymptotic behaviour for some classes of Jacobi matrices. This asymptotic and spectral analysis is provided in Section 2. The main two classes studied here are Jacobi matrices with zero diagonals and with weights being some perturbations of the sequence n^α.

In the first considered case $0<\alpha\leqslant 1$ and the perturbation is periodic. The most interesting result is the appearance of the gap in the absolutely continuous spectrum for the even period of the perturbation—this is a partial generalization of a result from [16].

The second one is the case with $\alpha>1$ (rapidly increasing weights) and with the perturbation of the form $c_n n^{\alpha-1}$, where $\{c_n\}$ is an even-periodic sequence. Under some extra assumptions we prove that the Jacobi operator is self-adjoint (which is false for the unperturbed weights!) and has only purely point spectrum.

In this section, we also show an example of a Jacobi matrix with divergent sequence of transfer matrices, which can be easily studied by asymptotic methods.

Section 1 is devoted to some abstract results which are used as "technical asymptotic tools" in Section 2. The basic result on asymptotic behaviour for difference equations was initiated by Poincaré and later improved by Perron (see e.g. [5,13]). The more subtle asymptotic behaviour of solutions of difference equations is provided by the so-called discrete Levinson theorem (see [1] and the above mentioned book [5] for details) being analogs of the Levinson theorem on the asymptotic behaviour of solutions of some ordinary differential equations—see [3]). The history of this kind of discrete asymptotic results is already long. For instance, one of the first such results was published (without proof) by Evgrafov [6], however, it contains some errors (see [1]). In Section 1 we also deal with some discrete versions of the Levinson theorem for a system of d linear difference equations. The main Lemma 1.1 is an analog of Levinson's fundamental result for the asymptotic integration of perturbations of diagonal differential systems (see [3]). The difference between Theorem 1.1 and the corresponding results from the literature (e.g. [1]) is that we formulate the assumptions guaranteeing the existence of *one* solution with the special asymptotic behaviour, while in the literature some assumptions, guaranteeing the existence of a base of such solutions are the most frequently formulated ones. Moreover, some of our assumptions have more general form. However, they are also more abstract, and thus we formulate also some consequences of this theorem, which can by applied more directly (Theorems 1.2–1.7). Especially, satisfactory results are obtained for D^1—diagonalizable systems (Theorem 1.4) and for $d=2$ (Theorems 1.6 and 1.7). One of the obtained results is a discrete version of the Hartman–Winter theorem [7] (Theorem 1.3).

The results of Section 2 illustrate the possibility of studying various families of Jacobi matrices from a common perspective and, in our view, with simpler proofs. Moreover, any new extension of the discrete Levinson-type theorem should allow to receive new applications to spectral analysis of Jacobi operators. We plan to study such extensions in future.

Notation. The set of integers is denoted by $\mathbb{Z}$, and $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let us fix $d \in \mathbb{N}$. By $M_d(K)$ we denote the set of $d \times d$ matrices with entries in K for $K = \mathbb{C}$ or $\mathbb{R}$. We fix an arbitrary norm $\|\cdot\|$ in $\mathbb{C}^d$ and we use the same symbol also for the induced operator norm in $M_d(\mathbb{C})$. For $s \in \{1, \ldots, d\}$ the sth standard base vector of $\mathbb{C}^d$ is denoted by e_s, if $v \in \mathbb{C}^d$, then the sth coordinate of v is denoted by v_s or $(v)_s$, and $\operatorname{diag}(v)$ or $\operatorname{diag}(v_1, \ldots, v_d)$ is the diagonal matrix from $M_d(\mathbb{C})$ with $v_1, \ldots, v_d$ being the successive diagonal entries. We shall use the following convention for products of matrices (or numbers): $\prod_{j=k}^{l} A(j)$ equals $A(l), \ldots, A(k)$ if $l > k$, if $l = k$ it equals A_k, and if $l < k$ it equals I (or 1).

For $N \in \mathbb{Z}$ and for $X = K$, K^d or $M_d(K)$ we denote by $l_N(X)$ the set of all sequences $x = \{x(n)\}_{n \geqslant N}$ with $x(n) \in X$. If $x \in l_N(X)$ and $p > 0$, then $x \in l_N^p(X)$ iff $\sum_{n=N}^{+\infty} \|x(n)\|^p < +\infty$, and $x \in D_N^1(X)$ iff $\sum_{n=N}^{+\infty} \|x(n+1) - x(n)\| < +\infty$, where in the case $X = K$ we put $\|\cdot\| = |\cdot|$. Moreover, $l(X) = \bigcup_{N \in \mathbb{Z}} l_N(X)$ and similarly for $l^p(X)$ and $D^1(X)$. We often omit '(X)' in all the above symbols when the choice of X is clear (e.g. we write l_N^p, D^1, l). We use the symbol $\rightarrow$ to denote the convergence of a sequence, and for a convergent $x = \{x(n)\} \in l(X)$ the symbol x_∞ usually denotes the limit of x (for matrix sequences the capitals are used, e.g. $A(n) \rightarrow A_\infty$). Observe that all D^1 sequences (also called *bounded variation sequences*) are convergent, and thus, for instance, the symbol A_∞ has sense for any $\{A(n)\} \in D^1(M_d(\mathbb{C}))$. The symbol $\operatorname{discr} A$ denotes the discriminant of the characteristic polynomial of 2×2 matrix A, i.e., $\operatorname{discr} A = (\operatorname{tr} A)^2 - 4 \det A$.

For $x \in \mathbb{R}$ the integer part of x is denoted by $\operatorname{Ent}(x)$, and the sign of x by $\operatorname{sgn}(x)$ (i.e., $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$ and $\operatorname{sgn}(0) = 0$).

1. Asymptotic behaviour of solutions of discrete linear systems

Let $\{A(n)\}_{n \geqslant n_0} \in l(M_d(\mathbb{C}))$, where $n_0 \in \mathbb{Z}$ is fixed, and consider the equation

$$x(n+1) = A(n)x(n) \quad \text{for } n \geqslant n_0. \tag{1.1}$$

Any sequence $x = \{x(n)\}_{n \geqslant n_0} \in l(\mathbb{C}^d)$ satisfying (1.1), is called *a solution of* (1.1). If $N \geqslant n_0$, and $x' = \{x'(n)\}_{n \geqslant N} \in l(\mathbb{C}^d)$ satisfies $x'(n+1) = A(n)x'(n)$ for $n \geqslant N$, then we call x' *a solution of* (1.1) *for* $n \geqslant N$. If we assume that

$$\det A(n) \neq 0 \quad \text{for } n \geqslant n_0, \tag{1.2}$$

then the space of all the solutions of (1.1) is d-dimensional, and any solution x is uniquely determined by the formula $x(n) = \prod_{j=n_0}^{n-1} A(j) x_{n_0}$, $n \geqslant n_0$, where x_{n_0} is an arbitrary vector from $\mathbb{C}^d$. Unfortunately, this formula usually does not give any direct asymptotic information about the solution.

There are many results concerning various kinds of asymptotic behaviour obtained for various assumptions on $\{A(n)\}_{n\geqslant n_0}$. The results presented here are some discrete versions of the Levinson theorem (the analogs of the Levinson theorem on the asymptotic behaviour of solutions of some ordinary differential equations—see [3]).

Let us first precise what kinds of asymptotic behaviour will be considered here.

Definition 1.1. Let $\gamma\in l_{n_0}(\mathbb{C})$ with $\gamma(n)\neq 0$, and let $x, v\in l_{n_0}(\mathbb{C}^d)$ be such that $x(n) = \gamma(n)v(n)$ for $n\geqslant n_0$. Then x

(a) *has the weak asymptotics* $\gamma(n)$ iff v is bounded and $\inf_{n\geqslant n_0} ||v(n)|| > 0$;
(b) *has the asymptotics* $\gamma(n)v_\infty$ iff $v(n)\to v_\infty$ and $v_\infty\neq 0$.

We also use the standard notation: x is $O(\gamma(n))$, when v is bounded for x, v and γ as above. Let us first consider the case of *perturbed diagonal systems*, that is assume that

$$A(n) = \Lambda(n) + R(n), \quad n\geqslant n_0, \tag{1.3}$$

where $\{\Lambda(n)\}_{n\geqslant n_0}, \{R(n)\}_{n\geqslant n_0}\in l(M_d(\mathbb{C}))$ and

$$\Lambda(n) = \mathrm{diag}(\lambda_1(n), \dots, \lambda_d(n)), \quad \lambda_s(n)\neq 0, \quad s = 1, \dots, d, \quad n\geqslant n_0. \tag{1.4}$$

For $s = 1, \dots, d$ and $n, n'\geqslant n_0$ define

$$\varphi_s(n, n') = \prod_{j=n}^{n'} \lambda_s(j). \tag{1.5}$$

Let us denote by (DIAG) the assumptions and notations (1.2)–(1.5).

Observe, that in the case $R(n)\equiv 0$ there exist solutions $x_1, \dots, x_d$ of (1.1) given by $x_s(n) = \varphi_s(n_0, n-1)e_s$ (and thus these solutions form a base of the space of all solutions of (1.1)). In particular, x_s has asymptotics $\varphi_s(n_0, n-1)e_s$. The natural question to ask is: what kind of more general assumptions can guarantee, at least partially, the similar asymptotic behaviour of a solution? To answer this question we first formulate a lemma. In the lemma we use $N_0\in\mathbb{Z}$ instead of n_0, since we shall chose appropriately large $N_0\geqslant n_0$ later.

Denote $P_s := \mathrm{diag}(e_s)$, $s = 1, \dots, d$. Let $\gamma\in l_{N_0}(\mathbb{C})$, $\gamma(n)\neq 0$. Suppose that (DIAG) holds with $n_0 = N_0$ and define

$$\alpha_s(n) = \sum_{j=N_0}^{n-1} \left|\frac{\varphi_s(j+1, n-1)}{\gamma(n)(\gamma(j))^{-1}}\right| ||P_sR(j)||, \tag{1.6}$$

$$\beta_s(n) = \sum_{j=n}^{+\infty} \left|\frac{\gamma(j)(\gamma(n))^{-1}}{\varphi_s(n, j)}\right| ||P_sR(j)|| \tag{1.7}$$

for $n\geqslant N_0$, $s = 1, \dots, d$.

We start with the following general result concerning asymptotic formula of a solution to (1.1).

Lemma 1.1. *Suppose that* (DIAG) *holds with* $n_0 = N_0$ *and that* $m \in \{1, \dots, d\}$, $S \subset \{1, \dots, d\}$ *are such that*

$$\sup_{n \geqslant N_0} \left(\sum_{s \in S} \alpha_s(n) + \sum_{s \notin S} \beta_s(n) \right) = c_1 < 1 \tag{1.8}$$

and

$$\sup_{n \geqslant N_0} \left| \frac{\varphi_m(N_0, n-1)}{\gamma(n)} \right| = c_2 < +\infty. \tag{1.9}$$

Then Eq. (1.1) *with* $n_0 = N_0$ *has a non-zero solution* x *being* $O(\gamma(n))$. *Moreover, this solution has a form*

$$x(n) = \gamma(n) \left[\frac{\varphi_m(N_0, n-1)}{\gamma(n)} e_m + r(n) \right], \quad n \geqslant N_0, \tag{1.10}$$

where

$$\|r(n)\| \leqslant \left(\sum_{s \in S} \alpha_s(n) + \sum_{s \notin S} \beta_s(n) \right) \frac{c_2 \|e_m\|}{1 - c_1}. \tag{1.11}$$

Proof. For $n, n' \geqslant N_0$ define $\Phi(n, n') = \prod_{j=n}^{n'} \Lambda(j)$. Let $X_\gamma = \{x \in l_{N_0}(\mathbb{C}^d) \colon \|x\|_\gamma < +\infty\}$, where $\|x\|_\gamma = \sup_{n \geqslant N_0} \frac{\|x(n)\|}{|\gamma(n)|}$. For $x \in X_\gamma$ we put

$$(K_s x)(n) = \sum_{j=N_0}^{n-1} \Phi(N_0, n-1) \Phi(N_0, j)^{-1} P_s R(j) x(j)$$

for $s \in S$, $n \geqslant N_0$ and similarly, for $s \notin S$, $n \geqslant N_0$

$$(L_s x)(n) = \sum_{j=n}^{+\infty} \Phi(N_0, n-1) \Phi(N_0, j)^{-1} P_s R(j) x(j).$$

Observe, that the series defining $(L_s x)(n)$ is convergent by (1.8). Moreover, we have

$$\frac{\|(K_s x)(n)\|}{|\gamma(n)|} \leqslant \alpha_s(n) \|x\|_\gamma, \quad \frac{\|(L_{s'} x)(n)\|}{|\gamma(n)|} \leqslant \beta_{s'}(n) \|x\|_\gamma \tag{1.12}$$

for $s \in S$, $s' \notin S$. Thus K_s and $L_{s'}$ can be treated as a bounded linear endomorphism of the Banach space $(X_\gamma, \|\cdot\|_\gamma)$. Define $D := \sum_{s \in S} K_s - \sum_{s \notin S} L_s$. By (1.12)

$$\frac{\|(Dx)(n)\|}{|\gamma(n)|} \leqslant \left(\sum_{s \in S} \alpha_s(n) + \sum_{s \notin S} \beta_s(n) \right) \|x\|_\gamma, \tag{1.13}$$

and therefore by (1.8) $\|D\|_\gamma \leqslant c_1$, where $\|\cdot\|_\gamma$ is used also to denote the operator norm induced by the norm $\|\cdot\|_\gamma$ in X_γ. Let now $b \in l_{N_0}(\mathbb{C}^d)$, $b(n) = \varphi_m(N_0, n-1) e_m$ for $n \geqslant N_0$. By (1.9) $b \in X_\gamma$ and $\|b\|_\gamma \leqslant c_2 \|e_m\|$. Consider the equation for $x \in X_\gamma$

$$(I - D)x = b. \tag{1.14}$$

Since $||D||_\gamma \leqslant c_1 < 1$ this equation has a unique solution $x = (I - D)^{-1} b$ and $x \neq 0$ since $b \neq 0$. On the other hand, by the definitions of K_s and L_s, since $\Phi(N, j)$ and P_s are diagonal (and thus they all commute) and since $\sum_{s=1}^{d} P_s = I \in M_d(\mathbb{C})$, a solution of (1.14) is a solution of (1.1) for $n_0 = N_0$. (Equality (1.14): $x = b + Dx$ is a kind of the variation of constants formula for (1.1): $x(n+1) = \Lambda(n)x(n) + R(n)x(n)$). The solution x is $O(\gamma(n))$ since $x \in X_\gamma$. Now taking $r(n) = \gamma(n)^{-1}(Dx)(n)$ and using (1.13) we obtain (1.11) since

$$||x||_\gamma \leqslant ||(I - D)^{-1}||_\gamma ||b||_\gamma \leqslant \frac{||b||_\gamma}{1 - ||D||_\gamma} \leqslant \frac{c_2 ||e_m||}{1 - c_1}. \qquad \square$$

We shall use the above lemma to prove some results on existence of solutions having asymptotics (in the sense of Definition 1.1) with the growth $\varphi_m(n_0, n-1)$, i.e., with the growth of one of the solutions of the unperturbed system (with $R(n) \equiv 0$). Before we formulate the first theorem, let us note the following simple fact.

Remark 1.1. Let $N_0 \geqslant n_0$. Suppose that (1.2) holds and that $\{\lambda(n)\}_{n \geqslant n_0} \in l(\mathbb{C})$, $\lambda(n) \neq 0$. If $x' \in l_{N_0}(\mathbb{C}^d)$ of the form $x'(n) = (\prod_{j=N_0}^{n-1} \lambda(j)) v'(n)$ is a solution of (1.1) for $n \geqslant N_0$, then there exists a solution $x \in l_{n_0}(\mathbb{C}^d)$ of (1.1) having the form $x(n) = (\prod_{j=n_0}^{n-1} \lambda(j)) v(n)$, with $v(n) = v'(n)$ for $n \geqslant N_0$.

Assume (DIAG) and for $n \geqslant N_0 \geqslant n_0$, $m, s \in \{1, \dots, d\}$ define

$$\alpha_{ms}(N_0, n) = \sum_{j=N_0}^{n-1} \left| \frac{\varphi_s(j, n-1)}{\varphi_m(j, n-1)} \right| \frac{||P_s R(j)||}{|\lambda_s(j)|}, \tag{1.15}$$

$$\beta_{ms}(n) = \sum_{j=n}^{+\infty} \left| \frac{\varphi_m(n, j-1)}{\varphi_s(n, j-1)} \right| \frac{||P_s R(j)||}{|\lambda_s(j)|}. \tag{1.16}$$

Theorem 1.1. *Assume* (DIAG) *and let* $m \in \{1, \dots, d\}$, $S \subset \{1, \dots, d\}$, *and* $N_0 \geqslant n_0$.

(a) *If*

$$\sup_{n \geqslant N_0} \left(\sum_{s \in S} \alpha_{ms}(N_0, n) + \sum_{s \notin S} \beta_{ms}(n) \right) = c < 1, \tag{1.17}$$

then Eq. (1.1) *has a non-zero solution x being $O(\varphi_m(n_0, n-1))$. If moreover $c < \frac{1}{2}$, then x has the weak asymptotics $\varphi_m(n_0, n-1)$.*

(b) *If*

$$\forall_{s \in S} \quad \lim_{n \to +\infty} \alpha_{ms}(N_0, n) = 0 \quad \textit{and} \quad \forall_{s \notin S} \quad \lim_{n \to +\infty} \beta_{ms}(n) = 0, \tag{1.18}$$

then Eq. (1.1) *has a solution with the asymptotics $\varphi_m(n_0, n-1)e_m$.*

Proof. To prove version (a) we can apply Lemma 1.1 with $\gamma(n) = \varphi_m(N_0, n-1)$, since then $\alpha_s(n) = \alpha_{ms}(N_0, n)$, $\beta_s(n) = \beta_{ms}(n)$ for $n \geq N_0$ and $c_1 = c$, $c_2 = 1$. Thus using Remark 1.1, we obtain the existence of a non-zero $O(\varphi_m(n_0, n-1))$ solution x such that $x(n) = \varphi_m(n_0, n-1)v(n)$ for $n \geq n_0$, where $v(n) = e_m + r(n)$ with $||r(n)|| \leq \frac{c||e_m||}{1-c}$ for $n \geq N_0$ due to (1.10) and (1.11). But if $c < \frac{1}{2}$, then $\frac{c}{1-c} < 1$, and thus x has the weak asymptotics $\varphi_m(n_0, n-1)$. To prove version (b) observe first that from (1.18) and from the fact that

$$\alpha_{ms}(N_0', n) \leq \alpha_{ms}(N_0, n) \quad \text{for } n \geq N_0' \geq N_0 \tag{1.19}$$

it follows that (1.17) holds with $c \leq \frac{1}{7}$ and with some $N_0' \geq N_0$ in the place of N_0. Thus proceeding as in the proof of the version (a), we find a solution x such that $x(n) = \varphi_m(n_0, n-1)(e_m + r(n))$, where $||r(n)|| \leq \frac{7}{6}||e_m||(\sum_{s \in S} \alpha_{ms}(N_0, n) + \sum_{s \notin S} \beta_{ms}(n))$ for $n \geq N_0'$ due to (1.11) and (1.19). Therefore, $r(n) \to 0$ by (1.18) and hence x has the asymptotics $\varphi_m(n_0, n-1)e_m$. □

The proofs of the discrete Levinson theorems from [1,5], as well as the proof of their primary continuous version [3] are based on the variation of constants formula and on the method of successive approximation, which is in fact similar to the inversion of the $(I - D)$ operator used here.

As an illustration of the above considerations let us show that Benzaid–Lutz theorem (see [1,5]) can be easily proved by the use of Theorem 1.1.

Theorem 1.2. *Assume* (DIAG) *and for $s, t \in \{1, \ldots, d\}$ set*

$$C_{st} = \sup_{n,n' \geq n_0} \left| \frac{\varphi_s(n, n')}{\varphi_t(n, n')} \right|. \tag{1.20}$$

Suppose that for any $s, t \in \{1, \ldots, d\}$ either

$$\lim_{n \to +\infty} \left| \frac{\varphi_s(n_0, n)}{\varphi_t(n_0, n)} \right| = 0 \quad \textit{and} \quad C_{st} < +\infty \tag{1.21}$$

or

$$C_{ts} < +\infty. \tag{1.22}$$

If for any $s \in \{1, \ldots, d\}$

$$\sum_{j=n_0}^{+\infty} \frac{||R(j)||}{|\lambda_s(j)|} < +\infty, \tag{1.23}$$

then there exists a base $x_1, \ldots, x_d$ of the space of solutions of (1.1) *such that x_m has the asymptotics $\varphi_m(n_0, n-1)e_m$ for $m = 1, \ldots, d$.*

Proof. Let us fix $m \in \{1, \ldots, d\}$ and choose $N_0 = n_0$ and $S := \{s \in \{1, \ldots, d\}$: Eq. (1.21) holds with $t = m\}$. Thus if $s \notin S$ then $C_{ms} < +\infty$, and $\lim_{n \to +\infty} \beta_{ms}(n) =$

0 due to (1.23). Let $s \in S$ and define

$$f_n^{[s]}(j) = \chi_n(j)\left|\frac{\varphi_s(n_0, n-1)}{\varphi_m(n_0, n-1)}\right|\left|\frac{\varphi_m(n_0, j-1)}{\varphi_s(n_0, j-1)}\right|\frac{||P_s R(j)||}{|\lambda_s(j)|}$$

for $n, j \geqslant n_0$, where $\chi_n(\cdot)$ is the characteristic function of the set $\{n_0, \dots, n-1\}$. By (1.21) $\lim_{n\to+\infty} f_n^{[s]}(j) = 0$ for any $j \geqslant n_0$. Moreover,

$$f_n^{[s]}(j) \leqslant \left|\frac{\varphi_s(j, n-1)}{\varphi_m(j, n-1)}\right|\frac{||P_s R(j)||}{|\lambda_s(j)|} \leqslant C_{sm}\frac{||R(j)||}{|\lambda_s(j)|}$$

for $n, j \geqslant n_0$. Thus, using the Lebesgue theorem on majorated convergence we obtain $\lim_{n\to+\infty} \alpha_{m,s}(n_0, n) = 0$, since $\alpha_{m,s}(n_0, n) = \sum_{j=n_0}^{+\infty} f_n^{[s]}(j)$. Therefore, by Theorem 1.1, there exists a solution x_m with the asymptotics $\varphi_m(n_0, n-1)e_m$. Now $x_1, \dots, x_d$ is a basis, since the system $e_1, \dots, e_d$ and thus also $x_1(n), \dots, x_d(n)$ for large n are linearly independent. $\square$

The typical situation when the assumptions of the above theorem are satisfied is the *simple modulus limit case*, that is the case when (assuming (DIAG))

$$\Lambda(n) \to \Lambda_\infty = \operatorname{diag}(\lambda_{1\infty}, \dots, \lambda_{d\infty})$$
$$\text{with } |\lambda_{t\infty}| \neq |\lambda_{s\infty}| \quad \text{for } s \neq t,\ \lambda_{t\infty} \neq 0,\ \ s, t = 1, \dots, d \tag{1.24}$$

and

$$\{R(n)\}_{n \geqslant n_0} \in l^1. \tag{1.25}$$

The following two-dimensional result can be immediately obtained from the above theorem.

Corollary 1.1. *Let* $d = 2$. *Assume* (DIAG) *and* (1.23) *for* $s = 1$ *or* 2. *If there exist* $0 < \delta, M < +\infty$ *such that*

$$\forall_{n,n' \geqslant n_0} \quad \delta \leqslant \left|\frac{\varphi_1(n, n')}{\varphi_2(n, n')}\right| \leqslant M,$$

then the assertion of Theorem 1.2 *holds.*

The finishing perturbed diagonal system result considered here may be treated as a discrete version of the Hartman–Winter theorem [7]. It refers again to the case $d = 2$ with the weaker then (1.25) assumption on the perturbation. A more general discrete analogue of the Hartman–Winter theorem was also given by Benzaid–Lutz [1, Corollary 3.4]. For the reader's convenience we present our version together with its simple proof.

Theorem 1.3. *Let* $d=2$. *Assume* (DIAG) *and* (1.24). *Suppose that* $R(n)=R_{\mathrm{D}}(n)+R_{\mathrm{AD}}(n)$, *where*

$$R_{\mathrm{D}}(n)=\begin{pmatrix} r_{11}(n), & 0 \\ 0, & r_{22}(n) \end{pmatrix}, \quad R_{\mathrm{AD}}(n)=\begin{pmatrix} 0, & r_{12}(n) \\ r_{21}(n), & 0 \end{pmatrix},$$

and $\{R_{\mathrm{AD}}(n)\}_{n\geqslant n_0}\in l^2$, $\{R_{\mathrm{D}}(n)\}_{n\geqslant n_0}\in l^1$. *Then there exists a base* x_1, x_2 *of the space of solutions of* (1.1) *such that* x_m *has the asymptotics* $\varphi_m(n_0, n-1)e_m$ *for* $m=1,2$.

To prove this theorem we need the following lemma.

Lemma 1.2. *Suppose that* $\{b(n)\}_{n\geqslant N}\in l^p(\mathbb{C})$, *where* $p\geqslant 1$, *and* $\{a(n)\}_{n\geqslant N}\in l(\mathbb{C})$, $a(n)\to a_\infty$ *with* $|a_\infty|\neq 1$, $a(n)\neq 0$. *Then the equation*

$$u(n+1)=a(n)u(n)+b(n), \quad n\geqslant N \tag{1.26}$$

has a solution $u=\{u(n)\}_{n\geqslant N}\in l^p(\mathbb{C})$.

Proof. Suppose first that $|a_\infty|>1$ and observe that u given by

$$u(n):=-\sum_{k=n}^{+\infty} b(k)\left(\prod_{s=n}^{k} a(s)\right)^{-1}, \quad n\geqslant N$$

is a well-defined solution of (1.26). Choose $0<\alpha<1$ and $N_0\geqslant N$ such that $|a(n)|^{-1}\leqslant\alpha$ for $n\geqslant N_0$, and define two sequences on $\mathbb{Z}$

$$\tilde{b}(n):=\begin{cases} 0 & \text{for } n<N_0, \\ |b(n)| & \text{for } n\geqslant N_0, \end{cases} \quad \tilde{\alpha}(n):=\begin{cases} \alpha^{-n} & \text{for } n\leqslant 0, \\ 0 & \text{for } n>0. \end{cases}$$

We have $|u(n)|\leqslant(\tilde{b}*\tilde{\alpha})(n)$ for $n\geqslant N_0$ and thus, by Young's inequality, $u\in l^p(\mathbb{C})$ since $\tilde{\alpha}$ is an l^1 sequence on $\mathbb{Z}$. Now suppose that $|a_\infty|<1$ and consider u defined by

$$u(n):=\sum_{k=N_0}^{n-1} b(k)\prod_{s=k+1}^{n-1} a(s), \quad n\geqslant N_0,$$

where $N_0\geqslant N$, $0<\alpha<1$ and $|a(n)|\leqslant\alpha$ for $n\geqslant N_0$. This is an l^p solution of (1.26) for $n\geqslant N_0$, since similarly to the first case we have $|u(n+1)|\leqslant(\tilde{b}*\tilde{\tilde{\alpha}})(n)$ for $n\geqslant N_0$, where

$$\tilde{\tilde{\alpha}}(n):=\begin{cases} \alpha^{n} & \text{for } n\geqslant 0, \\ 0 & \text{for } n<0. \end{cases} \qquad \square$$

Proof of Theorem 1.3. We apply here the idea of Harris–Lutz method (see [9]). Suppose that $\{Q(n)\}_{n\geqslant n_0}\in l^2(M_2(\mathbb{C}))$, $Q(n)=\begin{pmatrix} 0, & q_1(n) \\ q_2(n), & 0 \end{pmatrix}$. Then there exists $N\geqslant n_0$ such that $(I+Q(n))^{-1}=(1+\varepsilon(n))(I-Q(n))$ for $n\geqslant N$, where $\{\varepsilon(n)\}_{n\geqslant N}\in l^1$. Let $x=\{x(n)\}_{n\geqslant N}\in l(\mathbb{C}^2)$ and define $z=\{z(n)\}_{n\geqslant N}$ by the formula $z(n)=(I+$

$Q(n))^{-1}x(n)$. Thus x is a solution of (1.1) for $n \geqslant N$ iff z is a solution of the equation

$$z(n+1) = [\Lambda(n) + S'(n) + S''(n)]z(n), \quad n \geqslant N, \tag{1.27}$$

where $\{S''(n)\}_{n \geqslant N} \in l^1(M_2(\mathbb{C}))$ and

$$S'(n) = \Lambda(n)Q(n) - Q(n+1)\Lambda(n) + R_{\mathrm{AD}}(n). \tag{1.28}$$

Assume that we have chosen $\{Q(n)\}_{n \geqslant N}$ such that $S'(n) = 0$ for all $n \geqslant N$. Then Eq. (1.27) has a form satisfying the assumptions (DIAG), (1.24) and (1.25), and consequently, by Theorem 1.2 this equation has a base z_1, z_2 of the space of solutions of this equations, such that z_m has the asymptotics $\varphi_m(n_0, n-1)e_m$ for $m = 1, 2$. Hence, the same is true for Eq. (1.1), because we can take x_1, x_2 defined by $x_m(n) = (I + Q(n))z_m(n)$, $m = 1, 2$, as the base for $n \geqslant N$ and then we can use Remark 1.1 to extrapolate this base for $n \geqslant n_0$. It remains to prove that there exists some $\{Q(n)\}_{n \geqslant n_0}$ satisfying the above conditions. Observe that the condition $S'(n) \equiv 0$ can be written in the form of two independent scalar equations for $\{q_1(n)\}_{n \geqslant N}$ and $\{q_2(n)\}_{n \geqslant N}$:

$$q_1(n+1) = \frac{\lambda_1(n)}{\lambda_2(n)} q_1(n) + r_{12}(n), \quad q_2(n+1) = \frac{\lambda_2(n)}{\lambda_1(n)} q_2(n) + r_{21}(n),$$

$n \geqslant N$. But by Lemma 1.2 the both equations have solutions in l^2. $\square$

We shall consider now the more general case when the unperturbed system is not necessarily diagonal. Thus, in the place of (1.3) we assume that

$$A(n) = V(n) + R(n), \quad n \geqslant n_0, \tag{1.29}$$

where $\{V(n)\}_{n \geqslant n_0}, \{R(n)\}_{n \geqslant n_0} \in l(M_d(\mathbb{C}))$. The diagonality of $\{V(n)\}_{n \geqslant N}$ will be replaced by a weaker assumption, namely, by *D^1-diagonalizability*.

Definition 1.2. Suppose that $\{V(n)\}_{n \geqslant n_0}, \{\Lambda(n)\}_{n \geqslant n_0} \in l(M_d(\mathbb{C}))$ and that all $\Lambda(n)$ are diagonal. Then $\{V(n)\}_{n \geqslant n_0}$ *is D^1-diagonalizable to* $\{\Lambda(n)\}_{n \geqslant n_0}$ iff there exist $n_1 \geqslant n_0$ and $\{T(n)\}_{n \geqslant n_1} \in D^1(M_d(\mathbb{C}))$ such that $\det T_\infty \neq 0$, and for $n \geqslant n_1$

$$V(n) = T(n)\Lambda(n)(T(n))^{-1}. \tag{1.30}$$

Moreover, if the above holds, then T_∞ is called *a diagonalizing limit*.

Observe that in this definition $\Lambda(n)$ has to be a diagonal form of $V(n)$ only for large n. Observe also that the existence of the limit T_∞ follows from the assumption $\{T(n)\}_{n \geqslant n_1} \in D^1$.

Let us denote by (D^1-DIAG) the following assumptions and notations: (1.2) and (1.29), $\{V(n)\}_{n \geqslant n_0}$ is D^1-diagonalizable to $\{\Lambda(n)\}_{n \geqslant n_0}$ with a diagonalizing limit T_∞, (1.4) and (1.5).

We can formulate now the main theorem for D^1-diagonalizable case using assumptions of the form similar to these used in Benzaid–Lutz theorem.

Theorem 1.4. *Assume that* (D^1-DIAG) *holds, and that for some* $m\in\{1,\dots,d\}$ *there exists* $S\subset\{1,\dots,d\}$ *such that*

$$\forall_{s\in S}\ \lim_{n\to+\infty}\left|\frac{\varphi_s(n_0,n)}{\varphi_m(n_0,n)}\right|=0 \quad \textit{and} \quad C_{sm}<+\infty \tag{1.31}$$

and

$$\forall_{s\notin S} C_{ms}<+\infty, \tag{1.32}$$

where C_{st} *is defined by* (1.20). *If for any* $s,t\in\{1,\dots,d\}$

$$\sup_{n\geqslant n_0}\left|\frac{\lambda_s(n)}{\lambda_t(n)}\right|<+\infty \tag{1.33}$$

and

$$\sum_{j=n_0}^{+\infty}\frac{||R(j)||}{|\lambda_m(j)|}<+\infty, \tag{1.34}$$

then Eq. (1.1) *has a solution with the asymptotics* $\varphi_m(n_0,n-1)T_\infty e_m$.

Proof. Suppose that (1.30) holds for $n\geqslant N\geqslant n_0$. Let $x=\{x(n)\}_{n\geqslant N}\in l(\mathbb{C}^d)$ and define $z=\{z(n)\}_{n\geqslant N}$ by $z(n)=(T(n))^{-1}x(n)$. Thus x is a solution of (1.1) for $n\geqslant N$ iff z is a solution of the equation

$$z(n+1)=(\Lambda(n)+R'(n))z(n),\quad n\geqslant N, \tag{1.35}$$

where

$$R'(n)=(T(n))^{-1}R(n)T(n)+S(n)(V(n)+R(n))T(n)$$

with $S(n)=(T(n+1))^{-1}-(T(n))^{-1}$. We have $\{(T(n))^{-1}\}_{n\geqslant N}\in D^1$, because $\det T_\infty\neq 0$, and thus $\{S(n)\}_{n\geqslant N}\in l^1$. By (1.33) and (1.30) for any $s\in\{1,\dots,d\}, n\geqslant N$ we obtain

$$\begin{aligned}\frac{||R'(n)||}{|\lambda_s(n)|}&\leqslant K_1\frac{||R(n)||}{|\lambda_m(n)|}+K_2||S(n)||\left(\frac{||V(n)||}{|\lambda_m(n)|}+\frac{||R(n)||}{|\lambda_m(n)|}\right)\\&\leqslant K_3\frac{||R(n)||}{|\lambda_m(n)|}+K_4||S(n)||\left|\left|\operatorname{diag}\left(\frac{\lambda_1(n)}{\lambda_m(n)},\dots,\frac{\lambda_d(n)}{\lambda_m(n)}\right)\right|\right|\\&\leqslant K_5\left(\frac{||R(n)||}{|\lambda_m(n)|}+||S(n)||\right)\end{aligned}$$

for some constants $K_1,\dots,K_5$. Hence by (1.34) $\sum_{n\geqslant N}^{+\infty}\frac{||R'(n)||}{|\lambda_s(n)|}<+\infty$, for any s. Now by (1.31) and (1.32), using exactly the same arguments as in the proof of Theorem 1.2 we can check that Eq. (1.35) has a form satisfying the assumptions of Theorem 1.1(b) (with N instead of n_0 and $N_0=N$). Thus this equation has a solution z with the asymptotics $\varphi_m(n_0,n-1)e_m$. To finish the proof it is enough to use $T(n)\to T_\infty$ and Remark 1.1. $\square$

A simple consequence of Theorem 1.4 is the following result.

Corollary 1.2. *Assume that* (D^1-DIAG) *holds. Suppose that* $m \in \{1, \dots, d\}$ *is such that*

$$\exists_{n_1 \geqslant n_0,\ \delta > 0} \quad \forall_{s \neq m,\ n \geqslant n_1} \quad \delta \leqslant \left| \frac{\lambda_m(n)}{\lambda_s(n)} \right| \leqslant 1, \tag{1.36}$$

and that (1.34) *holds. Then Eq.* (1.1) *has a solution with the asymptotics* $\varphi_m(n_0, n-1) T_\infty e_m$.

Proof. We can apply Theorem 1.4 taking $S := \phi$. $\square$

The remaining part of this section consists of various consequences of Theorem 1.4 which are important for studies of difference operators, especially of Jacobi operators. The first theorem given below could be treated as the discretization of the differential Levinson theorem [3, Theorem 8.1], but assumption (iii) has the form which is rather similar to the main assumption of the Poincaré and Perron theorems. Although Theorem 1.5 will not be applied in the present paper, it is stated and proved here due to its usefulness in spectral analysis of some Jacobi operators, see [11,12]. This result was already formulated by Janas and Naboko [11] (for $d = 2$ and without proof; the similar result has been also formulated before by Evgrafov [6], but his formulation contains serious mistakes—see [1, Section 4]). Our formulation seems to be a new result, though it is similar to a result from [1].

Theorem 1.5. *Assume that* $A(n)$ *has form* (1.29) *and that following conditions are satisfied*:

(i) $\det A(n), \det V(n) \neq 0$ *for* $n \geqslant n_0$,
(ii) $\{V(n)\}_{n \geqslant n_0} \in D^1$,
(iii) $\lim_{n \to +\infty} V(n) =: V_\infty$ *has d non-zero eigenvalues* $\lambda_{1\infty}, \dots, \lambda_{d\infty}$ *having distinct absolute values*,
(iv) $\{R(n)\}_{n \geqslant n_0} \in l^1$.

Then there exists a base $x_1, \dots, x_d$ *of the space of solutions of* (1.1) *such that* x_m *has the asymptotics* $(\prod_{j=n_0}^{n-1} \lambda_m(j)) v_{m\infty}$, *where* $\lambda_m(n) \to \lambda_{m\infty}$, $\lambda_m(n)$ *is an eigenvalue of* $V(n)$, *and* $v_{m\infty}$ *is an eigenvector of* V_∞ *for* $\lambda_{m\infty}$, *for* $m = 1, \dots, d$.

We need the following D^1-diagonalizability criterion (also being a discrete version of a result from [3]) to prove this theorem.

Lemma 1.3. *Suppose that* $\{V(n)\}_{n \geqslant n_0} \in D^1(M_d(\mathbb{C}))$ *and* $\lim_{n \to +\infty} V(n) =: V_\infty$ *has d distinct eigenvalues* $\lambda_{1\infty}, \dots, \lambda_{d\infty}$. *Let for* $m = 1, \dots, d$ $\{\lambda_m(n)\}_{n \geqslant n_0}$ *be such that* $\lambda_m(n)$ *is an eigenvalue of* $V(n)$ *and* $\lambda_m(n) \to \lambda_{m\infty}$. *Define* $\Lambda(n) := \operatorname{diag}(\lambda_1(n) \dots, \lambda_d(n))$ *and* $\Lambda_\infty := \operatorname{diag}(\lambda_{1\infty} \dots, \lambda_{d\infty})$. *Then* $\{V(n)\}_{n \geqslant n_0}$ *is* D^1*-diagonalizable to* $\{\Lambda(n)\}_{n \geqslant n_0}$ *with a diagonalizing limit* T_∞ *such that* $V_\infty = T_\infty \Lambda_\infty T_\infty^{-1}$.

The proof of this lemma follows immediately from [10, Lemma 1.7].

Note that the sequences $\{\lambda_m(n)\}_{n\geqslant n_0}$ from the lemma exist and are unique for large n (i.e. two sequences of eigenvalues of $V(n)$ with the same limit $\lambda_{m\infty}$ have to be equal for n large enough).

Proof of Theorem 1.5. Define $\Lambda(n)$ and Λ_∞ as in the lemma. Thus all the assumptions of (D^1-DIAG) are satisfied. Let $m\in\{1,\dots,d\}$ and set $S=\{s\in\{1,\dots,d\}: |\lambda_{s\infty}|<|\lambda_{m\infty}|\}$. Condition (1.31) holds since $\Lambda(n)\to\Lambda_\infty$. If $s\notin S$ then for $s=m$ we have $C_{ms}=1$, and for $s\neq m$ we have $|\lambda_{s\infty}|>|\lambda_{m\infty}|$ by (iii), which gives $C_{ms}<+\infty$. Thus (1.32) holds and (1.33) and (1.34) are obvious, since $\lambda_{s\infty}\neq 0$ for any s and (iv) holds. Hence, by Theorem 1.4, for any m Eq. (1.1) has a solution x_m with the asymptotics $(\prod_{j=n_0}^{n-1}\lambda_m(j))v_{m\infty}$, where $v_{m\infty}=T_\infty e_m\neq 0$, and by Lemma 1.3 $V_\infty v_{m\infty}=T_\infty\Lambda_\infty e_m=\lambda_{m\infty}v_{m\infty}$. □

The most important in Theorem 1.5 were the assumptions that $\{V(n)\}_{n\geqslant n_0}\in D^1$, $\{R(n)\}_{n\geqslant n_0}\in l^1$ and that the eigenvalues of V_∞ have distinct absolute values. Below we show some new two-dimensional results for the cases when the eigenvalues of V_∞ may have equal absolute values. Nevertheless, all these results refer to the case when V_∞ is diagonalizable. In the first result this diagonalizability follows from the condition $\operatorname{discr} V_\infty<0$, and the eigenvalues are distinct, but have *the same* absolute value since V_∞ is real.

Theorem 1.6. *Let* $d=2$. *Assume* (1.29) *and conditions* (i), (ii), (iv) *of Theorem* 1.5. *If* $V(n)\in M_2(\mathbb{R})$ *for* $n\geqslant n_0$ *and* $\operatorname{discr} V_\infty<0$, *where* $V_\infty:=\lim_{n\to+\infty}V(n)$, *then the assertion of Theorem* 1.5 *holds.*

Proof. Observe first that the condition $V_\infty\in M_2(\mathbb{R})$ and $\operatorname{discr} V_\infty<0$ follow that $\lambda_{2\infty}=\bar{\lambda}_{1\infty}\notin\mathbb{R}$. Thus $\lambda_{1\infty}\neq\lambda_{2\infty}$ and the assumptions of Lemma 1.3 are satisfied. Moreover, $\operatorname{discr} V(n)<0$ for large n, which gives $|\lambda_1(n)|=|\lambda_2(n)|$. Hence for any $m\in\{1,2\}$ condition (1.36) holds. Condition (1.34) also holds, because $\lambda_{m\infty}\neq 0$ and $\{R(n)\}_{n\geqslant n_0}\in l^1$. Hence the assertion follows immediately from Corollary 1.2. □

Note that if we assume in the above theorem that $\operatorname{discr} V_\infty>0$ (instead of $\operatorname{discr} V_\infty<0$), then Theorem 1.5 can usually be used.

In the last theorem we investigate two cases when $V_\infty=aI$.

Theorem 1.7. *Let* $d=2$. *Assume* (1.29) *with* $\det A(n),\det V(n)\neq 0$ *for* $n\geqslant n_0$, *and suppose that* $V(n)$ *has the form*

$$V(n)=a(n)I+p(n)S(n),\quad n\geqslant n_0, \tag{1.37}$$

where

(i) $\{S(n)\}_{n\geqslant n_0}\in D^1(M_2(\mathbb{R}))$;
(ii) $\{a(n)\}_{n\geqslant n_0},\{p(n)\}_{n\geqslant n_0}\in l(\mathbb{R})$, $a(n)\to a_\infty\neq 0$, $p(n)\to 0$;
(iii) $\{R(n)\}_{n\geqslant n_0}\in l^1(M_2(\mathbb{C})$.

Let $S_\infty := \lim_{n\to+\infty} S(n)$ *and consider two cases*:

(a) $\operatorname{discr} S_\infty < 0$; (b) $\operatorname{discr} S_\infty > 0$ *and* $\operatorname{sgn}(p(n))$ *is constant and non-zero, and denote by* $\mu_{1\infty}, \mu_{2\infty}$ *the eigenvalues of* S_∞, *where in case* (b) $\operatorname{sgn}(\frac{p(n)}{a_\infty})(\mu_{2\infty} - \mu_{1\infty}) > 0$. *For Eq.* (1.1) *there exists*

- *a base* $x_1,\ x_2$ *of the space of solutions in case* (a)
- *a non-zero solution* x_1 *in case* (b)

such that x_m *has the asymptotics*

$$\left(\prod_{j=n_0}^{n-1} (a(j) + p(j)\mu_m(j))\right) s_{m\infty}$$

for (a) $m = 1, 2$, *for* (b) $m = 1$, *where* $\mu_m(n) \to \mu_{m\infty}$, $\mu_m(n)$ *is an eigenvalue of* $S(n)$, *and* $s_{m\infty}$ *is an eigenvector of* S_∞ *for* $\mu_{m\infty}$.

Proof. In the both cases (a) and (b) $\mu_{1\infty} \neq \mu_{2\infty}$. Thus, we can apply Lemma 1.3 for $\{S(n)\}_{n\geqslant n_0}$ and using (1.37) we conclude that $\{V(n)\}_{n\geqslant n_0}$ is D^1-diagonalizable to $\{\Lambda(n)\}_{n\geqslant n_0}$, where $\Lambda(n) = \operatorname{diag}(\lambda_1(n), \lambda_2(n))$ with $\lambda_m(n) = a(n) + p(n)\mu_m(n)$, with a diagonalizing limit T_∞ such that $S_\infty = T_\infty \operatorname{diag}(\mu_{1\infty}, \mu_{2\infty}) T_\infty^{-1}$. We shall use now Corollary 1.2. By (ii) and (iii) (1.34) holds. In case (a) we obtain (1.36) for any choice of $m \in \{1, 2\}$ since $\overline{\lambda_1(n)} = \lambda_1(n)$ for large n (because $a(n), p(n) \in \mathbb{R}$). In case (b) for $m = 1$ (1.36) holds, since by (ii) $\frac{\lambda_1(n)}{\lambda_2(n)} \to 1$ ($\lambda_s(n) \neq 0$ since it is an eigenvalue of $V(n)$ and $\det V(n) \neq 0$) and for $\sigma := \operatorname{sgn}(\frac{p(n)}{a_\infty})$, and for n large enough

$$|\lambda_1(n)| = |a(n)|\left(1 + \left|\frac{p(n)}{a(n)}\right|\sigma\mu_1(n)\right) \leqslant |a(n)|\left(1 + \left|\frac{p(n)}{a(n)}\right|\sigma\mu_2(n)\right) = |\lambda_2(n)|. \qquad \square$$

2. Applications to Jacobi matrices

In this section, we consider some infinite Jacobi matrices of the form

$$\begin{pmatrix} 0 & b(1) & & & \\ b(1) & 0 & b(2) & & \\ & b(2) & 0 & b(3) & \\ & & b(3) & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

where $\{b(n)\}_{n\geqslant 1}$ has non-zero real terms (the so-called *weights*). We also consider corresponding operators J acting in the Hilbert space $l_1^2 = l_1^2(\mathbb{C})$ given by

$$Ju = \mathscr{J}u$$

for u from the maximal domain $D(J) = \{u \in l_1^2\colon\ \mathcal{J}u \in l_1^2\}$, where

$$(\mathcal{J}u)(n) = b(n-1)u(n-1) + b(n)u(n+1), \quad n \in \mathbb{N}$$

with the convention that $b(j), u(j) = 0$ for $j < 1$.

Let $\lambda \in \mathbb{R}$. The equation

$$(\mathcal{J}u)(n) = \lambda u(n), \quad n \geqslant \mathbf{2} \tag{2.1}$$

for a complex sequence $u = \{u(n)\}_{n \geqslant 1}$ we call *the generalized eigenequation for J and* λ. Note that this equation is "generalized" because we do not assume that $u \in l_1^2$, and also because we omit $n = 1$ in (2.1) (and thus usually a solution u is not an eigenvector of J, even when $u \in l_1^2$). Observe that the generalized eigenequation is a scalar second-order linear difference equation. It can be written in the equivalent $\mathbb{C}^2$ vector form of (1.1) type

$$\tilde{u}(n+1) = B(n)\tilde{u}(n), \quad n \geqslant 2, \tag{2.2}$$

where $\tilde{u}(n) = \binom{u(n-1)}{u(n)} \in \mathbb{C}^2$, and $B(n) \in M_2(\mathbb{R})$ are *transfer matrices*,

$$B(n) = \begin{pmatrix} 0 & 1 \\ -\dfrac{b(n-1)}{b(n)} & \dfrac{\lambda}{b(n)} \end{pmatrix}. \tag{2.3}$$

As we shall see, in the first two subsections of this section the transfer matrix sequence is convergent to E, which is given by

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and in the last one it is divergent. Our aim is to study the asymptotic behaviour of some solutions of the generalized eigenequation for J and $\lambda \in \mathbb{R}$, for some examples of J. In these examples the transfer matrices do not behave regularly enough to use the theory developed in the previous section directly for Eq. (2.2). Thus we consider the equation

$$x(n+1) = A(n)x(n), \quad n \geqslant 1 \tag{2.4}$$

for $x = \{x(n)\}_{n \geqslant 1} \in l(\mathbb{C}^2)$, where

$$A(n) = \prod_{j=0}^{T-1} B(nT + j), \tag{2.5}$$

and T is a certain natural number. Observe, that for any solution x of (2.4) there exists a unique solution u of the generalized eigenequation for J and λ satisfying $\tilde{u}(nT) = x(n)$ for $n \geqslant 1$. Moreover, the correspondence "$x \rightsquigarrow u$" is a linear isomorphism of the space of solution of (2.4) to the space of solutions of (2.1), and we have

$$u(nT+s) = \left(\prod_{j=0}^{s-1} B(nT+j)x(n) \right)_2 \quad \text{for } n \geqslant 1, \ \ s \geqslant 0. \tag{2.6}$$

We shall use the following spectral terminology. If H is a self-adjoint operator in a Hilbert space, then $\mathscr{H}_{\mathrm{ac}}(H)$, $\mathscr{H}_{\mathrm{sc}}(H)$, $\mathscr{H}_{\mathrm{pp}}(H)$ are the space of absolute continuity, of singular continuity, and the pure point space for H (i.e. the closure of the space spanned by all the eigenvectors for H), respectively; for a Borel subset $G \subset \mathbb{R}$, $\mathscr{H}_G(H)$ is the range of the spectral projection for H and G, and $\sigma_{\mathrm{ac}}(H)$, $\sigma_{\mathrm{pp}}(H)$, $\sigma_{\mathrm{sc}}(H)$ are the absolutely continuous, point, and the singular continuous spectrum of H, respectively. We say that H *is absolutely continuous* (*purely point*) *in* G iff $\mathscr{H}_G(H) \subset \mathscr{H}_{\mathrm{ac}}(H)$ ($\mathscr{H}_{\mathrm{pp}}(H)$), and we omit "in G" for $G = \mathbb{R}$.

2.1. A class of Jacobi matrices with periodically perturbed weights

In this subsection, the weight sequence is a periodic perturbation of the sequence n^α, i.e.,

$$b(n) = n^\alpha + c(n), \quad n \geqslant 1, \tag{2.7}$$

where $\alpha \in (0;1]$ and $c = \{c(n)\}_{n \in \mathbb{Z}}$ is a T-periodic real sequence, with a period $T \geqslant 2$, and $n^\alpha + c(n) \neq 0$ for $n \geqslant 1$. Since such weight sequence satisfies the Carleman condition, J is an unbounded self-adjoint operator (see e.g. [2]). We use here different methods for $T = 2L$ and for $T = 2L+1$ (with $L \in \mathbb{N}$). To study Eq. (2.4) we shall apply Theorems 1.7 and 1.6. It is convenient to introduce first a subclass of D^1, consisting of sequences with a special kind of asymptotics. Assume here that $N \in \mathbb{Z}$, and $X = \mathbb{R}, \mathbb{C}, \mathbb{C}^d$ or $M_d(\mathbb{C})$.

Definition 2.1. Let $x \in l_N(X)$. Then $x \in D_N^{1*}(X)$ iff there exist $x_\infty \in X$, $\gamma \in (0;1]$, $a \in D_N^1(X)$ and $r \in l_N^1(X)$ such that for $n \geqslant N$

$$x(n) = x_\infty + n^{-\gamma} a(n) + r(n). \tag{2.8}$$

We also set $D^{1*}(X) \coloneqq \bigcup_{N \in \mathbb{Z}} D_N^{1*}(X)$, and we use the symbols D_N^{1*} and D^{1*} when the choice of X is obvious. Let $x \in l(X)$, $x_\infty, a_\infty \in X$, $\gamma \in (0;1]$. We write

$$x(n) \approx x_\infty + n^{-\gamma} a_\infty \tag{2.9}$$

iff there exist $a \in D^1(X)$, $r \in l^1(X)$ such that (2.8) holds for large n and $a(n) \to a_\infty$ (thus x must be in $D^{1*}(X)$ in this case).

Note that x_∞, γ and a_∞ in (2.9) are not uniquely determined, but if (2.9) holds and at the same time

$$x(n) \approx x'_\infty + n^{-\gamma'} a'_\infty$$

with a_∞, $a'_\infty \neq 0$, then $x'_\infty = x_\infty$, $a'_\infty = a_\infty$, and $\gamma' = \gamma$. We shall use the following technical lemma.

Lemma 2.1. (a) *$D_N^{1*}(X)$ is a subalgebra of $D_N^1(X)$. Moreover, if $x'(n) \approx x'_\infty + n^{-\gamma'} a'_\infty$, $x''(n) \approx x''_\infty + n^{-\gamma''} a''_\infty$ then $(x'x'')(n) \approx x'_\infty x''_\infty + n^{-\gamma} a_\infty$, where $\gamma = \min(\gamma', \gamma'')$,*

$$a_\infty = \begin{cases} a'_\infty x''_\infty & \text{for } \gamma' < \gamma'', \\ a'_\infty x''_\infty + x'_\infty a''_\infty & \text{for } \gamma' = \gamma'', \\ x'_\infty a''_\infty & \text{for } \gamma' > \gamma''. \end{cases}$$

(b) *Suppose that $x \in D^{1*}(\mathbb{R})$, $x(n) \approx x_\infty + n^{-\gamma} a_\infty$, and $y(n) = f(x(n))$ for large n, where f is a real C^m function in a neighbourhood of x_∞, with $m > \gamma^{-1}$. Then $y(n) \approx f(x_\infty) + n^{-\gamma} f^{(1)}(x_\infty) a_\infty$.*

Proof. Condition (a) is immediate. To prove (b) we use Taylor's formula

$$f(x_\infty + t) = \sum_{j=0}^{m-1} \frac{f^{(j)}(x_\infty)}{j!} t^j + R(t), \quad R(t) = O(t^m), \ \ t \to 0$$

(note, that $m \geqslant 2$ since $\gamma \leqslant 1$), and for large n we obtain

$$\begin{aligned} y(n) &= f(x_\infty + n^{-\gamma} a(n) + r(n)) \\ &= \sum_{j=0}^{m-1} \frac{f^{(j)}(x_\infty)}{j!} \sum_{s=0}^{j} \binom{j}{s} n^{-s\gamma} (a(n))^s (r(n))^{j-s} + r_1(n) \\ &= \sum_{j=0}^{m-1} \frac{f^{(j)}(x_\infty)}{j!} n^{-j\gamma} (a(n))^j + r_2(n) = f(x_\infty) \\ &\quad + n^{-\gamma} \left[f^{(1)}(x_\infty) a(n) + \sum_{j=2}^{m-1} \frac{f^{(j)}(x_\infty)}{j!} n^{-(j-1)\gamma} (a(n))^j \right] + r_2(n), \end{aligned}$$

where $a \in D^1$, $r, r_1, r_2 \in l^1$. Thus, since D^1 is an algebra, we obtain the assertion of (b). $\square$

Now define

$$\sigma_\alpha = \begin{cases} 0 & \text{for } \alpha \in (0; 1), \\ \frac{1}{2} & \text{for } \alpha = 1. \end{cases}$$

We can prove the following result on asymptotic behaviour of the sequence $\{A(n)\}_{n \geqslant 1}$ given by (2.5).

Proposition 2.1. *$A(n) \approx A_\infty + n^{-\alpha} C_\lambda$, where*

$$A_\infty = \begin{cases} (-1)^L I & \text{for } T = 2L, \\ (-1)^L E & \text{for } T = 2L+1, \end{cases}$$

$$C_\lambda = \begin{cases} \dfrac{(-1)^L}{T^\alpha} \begin{pmatrix} -\rho + T\sigma_\alpha & -L\lambda \\ L\lambda & \rho + T\sigma_\alpha \end{pmatrix} & \text{for } T = 2L, \\ \dfrac{(-1)^L}{T^\alpha} \begin{pmatrix} L\lambda & \rho + 2L\sigma_\alpha \\ \rho - 2(L+1)\sigma_\alpha & (L+1)\lambda \end{pmatrix} & \text{for } T = 2L+1 \end{cases}$$

with

$$\rho = \sum_{j=0}^{2L-1} (-1)^j c(j) \quad \text{for } L = \mathrm{Ent}(T/2). \tag{2.10}$$

Proof. By (2.3), $B(nT+j) = E + \begin{pmatrix} 0 & 0 \\ \varepsilon_j(n) & \rho_j(n) \end{pmatrix}$ with

$$\varepsilon_j(n) = 1 - \left(\left(1 + \frac{j-1}{nT}\right)^\alpha + \frac{c(j-1)}{(nT)^\alpha} \right) \left((1 + \frac{j}{nT})^\alpha + \frac{c(j)}{(nT)^\alpha} \right)^{-1},$$

$$\rho_j(n) = \frac{\lambda}{(nT)^\alpha} \left(\left(1 + \frac{j}{nT}\right)^\alpha + \frac{c(j)}{(nT)^\alpha} \right)^{-1}.$$

Using Lemma 2.1 we obtain $\varepsilon_j(n) \approx 0 + n^{-\alpha} f_j$, where

$$f_j = T^{-\alpha}(c(j) - c(j-1)) + \begin{cases} 0 & \text{for } \alpha \in (0;1), \\ T^{-1} & \text{for } \alpha = 1, \end{cases} \tag{2.11}$$

and $\rho_j(n) \approx 0 + n^{-\alpha} \frac{\lambda}{T^\alpha}$. Thus

$$B(nT+j) \approx E + n^{-\alpha} \begin{pmatrix} 0 & 0 \\ f_j & \frac{\lambda}{T^\alpha} \end{pmatrix},$$

and by Lemma 2.1(a) and (2.5) $A(n) \approx E^T + n^{-\alpha} \tilde{C}_\lambda$, where

$$\tilde{C}_\lambda = \sum_{j=0}^{T-1} E^{T-1-j} \begin{pmatrix} 0 & 0 \\ f_j & \dfrac{\lambda}{T^\alpha} \end{pmatrix} E^j.$$

We also have $E^{2u} = (-1)^u I,\ E^{2u-1} = (-1)^{u-1} E$ and

$$E \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 \\ -b & a \end{pmatrix},$$

hence for $T = 2L$, using (2.11), we can compute

$$\tilde{C}_\lambda = \sum_{u=0}^{L-1} E^{2L-1-2u} \begin{pmatrix} 0 & 0 \\ f_{2u} & \frac{\lambda}{T^\alpha} \end{pmatrix} E^{2u} + \sum_{u=1}^{L} E^{2L-2u} \begin{pmatrix} 0 & 0 \\ f_{2u-1} & \frac{\lambda}{T^\alpha} \end{pmatrix} E^{2u-1}$$
$$= (-1)^{L-1} \left[\sum_{u=0}^{L-1} \begin{pmatrix} f_{2u} & \frac{\lambda}{T^\alpha} \\ 0 & 0 \end{pmatrix} + \sum_{u=1}^{L} \begin{pmatrix} 0 & 0 \\ -\frac{\lambda}{T^\alpha} & f_{2u-1} \end{pmatrix} \right]$$
$$= (-1)^{L-1} \begin{pmatrix} \sum_{u=1}^{L} f_{2u} & \frac{L\lambda}{T^\alpha} \\ -\frac{L\lambda}{T^\alpha} & \sum_{u=1}^{L} f_{2u-1} \end{pmatrix} = C_\lambda.$$

Similarly, for $T = 2L + 1$

$$\tilde{C}_\lambda = \sum_{u=0}^{L} E^{2L-2u} \begin{pmatrix} 0 & 0 \\ f_{2u} & \frac{\lambda}{T^\alpha} \end{pmatrix} E^{2u} + \sum_{u=1}^{L} E^{2L-2u+1} \begin{pmatrix} 0 & 0 \\ f_{2u-1} & \frac{\lambda}{T^\alpha} \end{pmatrix} E^{2u-1}$$
$$= (-1)^{L} \left[\sum_{u=0}^{L} \begin{pmatrix} 0 & 0 \\ f_{2u} & \frac{\lambda}{T^\alpha} \end{pmatrix} + \sum_{u=1}^{L} \begin{pmatrix} \frac{\lambda}{T^\alpha}, & -f_{2u-1} \\ 0 & 0 \end{pmatrix} \right] = C_\lambda. \qquad \square$$

Having the above result, we are ready to formulate and prove the theorem on asymptotic behaviour of generalized eigensolutions for J.

Theorem 2.1. *Consider the generalized eigenequation* (2.1) *for J determined by* (2.7) *and for $\lambda \in \mathbb{R}$ in the following three cases*:

(i) $T = 2L + 1$,
(ii) $T = 2L$ *and* $|\lambda| > \frac{|\rho|}{L}$,
(iii) $T = 2L$ *and* $|\lambda| < \frac{|\rho|}{L}$,

where ρ is given by (2.10). *For the equation there exists*:

– *a base $u_1,\ u_2$ of the space of solutions in cases* (i) *and* (ii),
– *a non-zero solution u_1 in case* (iii)

of the form

$$u_m(nT + s) = \left(\prod_{j=1}^{n-1} (1 + j^{-\alpha} \eta_m(j)) \right) \beta_m(nT + s), \tag{2.12}$$

$n \geqslant 0$, $s = 1, \dots, T$ ($m = 1, 2$ *for* (i) *and* (ii), *and* $m = 1$ for (iii)), *where* $\{\eta_m(n)\}_{n \geqslant 1}, \{\beta_m(k)\}_{k \geqslant 1} \in l(\mathbb{C})$ *are such that* $\eta_m(n) \to \eta_{m\infty}$, *and* $\lim_{k \to +\infty} \beta_m(k) - \tilde{\beta}_m(k) = 0$, *with* $\eta_{m\infty}$ *and with a* 4*-periodic sequence* $\{\tilde{\beta}_m(k)\}_{k \geqslant 1} \in l(\mathbb{C})$ *defined by the following formulas*:

for (i)

$$\eta_{m\infty} = \tfrac{1}{2}(-1)^{m+1} i T^{1-\alpha}\lambda + \sigma_\alpha, \quad \tilde{\beta}_m(k) = ((-1)^m i)^k;$$

for (ii)

$$\eta_{m\infty} = (-1)^{m+L} i T^{-\alpha}\sqrt{L^2\lambda^2 - \rho^2} + \sigma_\alpha,$$

$$\tilde{\beta}_m(k) = \begin{cases} (-1)^t d_m & \text{for } k = 2t, \\ (-1)^{t+1} & \text{for } k = 2t+1, \end{cases}$$

where $d_m = L\lambda((-1)^{m+L} i\sqrt{L^2\lambda^2 - \rho^2} - \rho)^{-1}$;
for (iii)

$$\eta_{1\infty} = -T^{-\alpha}\sqrt{\rho^2 - L^2\lambda^2} + \sigma_\alpha,$$

$$\tilde{\beta}_1(k) = \begin{cases} (-1)^t d_1 & \text{for } k = 2t, \\ (-1)^{t+1} d_1' & \text{for } k = 2t+1, \end{cases}$$

where

$$(d_1, d_1') = \begin{cases} (-L\lambda(\rho + \sqrt{\rho^2 - L^2\lambda^2})^{-1}, 1) & \text{for } \lambda \neq 0 \text{ or } \rho > 0, \\ (1, 0) & \text{for } \lambda = 0 \text{ and } \rho < 0. \end{cases}$$

Moreover, in cases (i) *and* (ii) *for any* $n \geqslant 1$ *we have*

$$\overline{\eta_1(n)} = \eta_2(n) \tag{2.13}$$

and

$$|1 + n^{-\alpha}\eta_m(n)| = \left|\frac{b(nT-1)}{b((n+1)T-1)}\right|^{1/2}(1 + r(n)) \tag{2.14}$$

with $\{r(n)\}_{n\geqslant 1} \in l^1$.

Proof. Consider Eq. (2.4). By (2.5) and by Proposition 2.1 we have (with the notation of the proposition) (1.2) and (1.29) with $n_0 = 1$, $\{R(n)\}_{n\geqslant 1} \in l^1$ and $V(n) = V_\infty + n^{-\alpha}S(n)$, where $V_\infty = A_\infty$, $\{S(n)\}_{n\geqslant 1} \in D^1$, and $S(n) \to S_\infty := C_\lambda$. Observe also that we can assume that $S(n) \in M_2(\mathbb{R})$, since $A(n), V_\infty, S_\infty \in M_2(\mathbb{R})$. Moreover, for $T = 2L+1$ $\operatorname{discr} V_\infty = -4$, and for $T = 2L$ $\operatorname{discr} S_\infty = 4T^{-2\alpha}(\rho^2 - L^2\lambda^2)$. Thus, to find asymptotics for some solutions of (2.4) we can use Theorem 1.6 for case (i) and Theorem 1.7(a) and (b) for cases (ii) and (iii), respectively. Using also

(2.6) we obtain the solutions of (2.1) of the form

$$u'_m(nT+s) = \left(\prod_{j=1}^{n-1}(1+j^{-\alpha}\eta_m(j))\right)\xi_m(n)\left(\prod_{j=0}^{s-1}B(nT+j)v_m(n)\right)_2 \tag{2.15}$$

for $n \geqslant 0$, $s = 1, \dots, T$, $m = 1, 2$ for (i) and (ii) and $m = 1$ for (iii), where:

(1) $$\xi_m(n) = \begin{cases} (i(-1)^{m+L})^{n-1} & \text{for (i)}, \\ (-1)^{L(n-1)} & \text{for (ii) and (iii)}, \end{cases}$$

(2) $$\lambda_m(n) := (1+n^{-\alpha}\eta_m(n))\begin{cases} i(-1)^{m+L} & \text{for } (i) \\ (-1)^L & \text{for } (ii) \text{ and } (iii) \end{cases}$$ are the eigenvalues of $V(n)$ for large n,

(3) $\lambda_m(n) \to (-1)^{m+L} i$ for (i), $(-1)^L \eta_m(n) \to \mu_{m\infty}$ for (ii) and (iii) with

$$\mu_{m\infty} = \begin{cases} (-1)^m i T^{-\alpha}\sqrt{L^2\lambda^2 - \rho^2} + (-1)^L \sigma_\alpha & \text{for (ii)}, \\ (-1)^{L+m} T^{-\alpha}\sqrt{\rho^2 - L^2\lambda^2} + (-1)^L \sigma_\alpha & \text{for (iii)}, \end{cases}$$

(4) $v_m(n) \to v_{m\infty}$ with

$$v_{m\infty} = \begin{cases} (1, (-1)^m i) & \text{for (i)}, \\ (1, L\lambda((-1)^{m+L} i\sqrt{L^2\lambda^2 - \rho^2} - \rho)^{-1}) & \text{for (ii)}, \end{cases}$$

and for (iii)

$$v_{1\infty} = \begin{cases} (1, -L\lambda(\rho + \sqrt{\rho^2 - L^2\lambda^2})^{-1}) & \text{for } \lambda \neq 0 \text{ or } \rho > 0, \\ (0, 1) & \text{for } \lambda = 0 \text{ and } \rho < 0. \end{cases}$$

Moreover, we can also assume that

(5) $\lambda_m(n) \neq 0$ and in cases (i) and (ii) $\overline{\lambda_1(n)} = \lambda_2(n)$ for any $n \geqslant 1$.

The linear independence of the two solutions of (2.4) follows the some for u'_1, u'_2 for (i) and (ii). In cases (ii) and (iii) the formula for $\eta_{m\infty}$ follows immediately from (3). In case (i), to obtain this formula, we can use (2) and (3) and the following perturbation theory result.

Lemma 2.2. *Suppose that* $\{U(n)\}_{n \geqslant n_0}$, $\{Q(n)\}_{n \geqslant n_0} \in l(M_2(\mathbb{C}))$ *satisfy*

$$U(n) = U_\infty + n^{-\alpha} Q(n), \quad n \geqslant n_0,$$

where $Q(n) \to Q_\infty$, $U_\infty = (U_{\infty kl})_{k,l=1,2}$, $Q_\infty = (Q_{\infty kl})_{k,l=1,2} \in M_2(\mathbb{C})$, $n_0 \geqslant 1$, $\alpha > 0$. *If* U_∞ *has two different eigenvalues* $\lambda_{1\infty}, \lambda_{2\infty}$, *then there exist* $\{\lambda_m(n)\}_{n \geqslant n_0} \in l(\mathbb{C})$ *for* $m = 1, 2$ *such that* $\lambda_m(n)$ *are eigenvalues of* $U(n)$ *for large* n *and* $\lambda_m(n) = \lambda_{m\infty} +$

$n^{-\alpha}q_m(n)$, *where*

$$q_m(n) \to q_{m\infty} := \frac{1}{2}\left[\frac{w_\infty}{2\lambda_{m\infty} - \operatorname{tr} U_\infty} + \operatorname{tr} Q_\infty\right], \quad m = 1, 2$$

with $w_\infty = (U_{\infty 11} - U_{\infty 22})(Q_{\infty 11} - Q_{\infty 22}) + 2(U_{\infty 12}Q_{\infty 21} + U_{\infty 21}Q_{\infty 12})$.

(We omit here the standard proof, based on the formula for roots of second-order equation.)

Since the $\lambda_m(n) - s$ from the lemma for $U(n) = V(n)$ and the ones from (2) are equal for n-large, thus, by Proposition 2.1 $(-1)^{m+L} i\eta_{m\infty} = q_{m\infty} = \frac{1}{2}(-1)^L T^{1-\alpha}\lambda + (-1)^{m+L} i\sigma_\alpha$, which proves the formula for $\eta_{m\infty}$ in case (i).

We have $B(n) \to E$, and therefore, by (4)

$$\xi_m(n)\left(\prod_{j=0}^{s-1} B(nT + j) v_m(n)\right)_2 = \xi_m(n)(E^s v_{m\infty})_2 + \varphi_m(nT + s),$$

where $\lim_{k\to+\infty} \varphi_m(k) = 0$. Moreover, $E^2 = -I$, and by (1) $\xi_m(n) = a_m((-1)^m i)^{nT}$, where a_m is a non-zero constant. Hence writing $k = nT + s$ we have

$$\begin{aligned}\xi_m(n)(E^s V_{m\infty})_2 &= a_m((-1)^m i)^{nT} \begin{cases} i^s (v_{m\infty})_2 & \text{for } s \in 2\mathbb{N}, \\ i^{s+1}(v_{m\infty})_1 & \text{for } s \notin 2\mathbb{N}, \end{cases} \\ &= a_m((-1)^m i)^k \begin{cases} (v_{m\infty})_2 & \text{for } s \in 2\mathbb{N}, \\ (-1)^m i (v_{m\infty})_1 & \text{for } s \notin 2\mathbb{N}. \end{cases}\end{aligned}$$

If $\tilde{\beta}_m$ is defined as in the assertion of the theorem, then by (4) we see that $\xi_m^{(n)}(E^s V_{m\infty})_2 = a'_m \tilde{\beta}_m(k)$ with a non-zero constant a'_m, for all cases (i)–(iii). Thus it remains only to prove (2.13) and (2.14) (for cases (i) and (ii)). The first formula is obvious by (2) and (5). To prove the second one observe, that by (2) and (5) for large n

$$|1 + n^{-\alpha}\eta_m(n)|^2 = |\det V(n)| = |\det A(n)||\det(I - (A(n))^{-1}R(n))|.$$

Therefore by (2.3) and (2.5), and the fact that $\{R(n)\}_{n\geqslant 1} \in l^1$ we obtain (2.14). □

Remark 2.1. (a) The restriction $\lambda \in \mathbb{R}$ is necessary to apply Theorems 1.6 and 1.7, since $V(n) \in M_2(\mathbb{R})$ is an important assumption for the both of them. We do not analyse also the points $\lambda = \pm\frac{\varrho}{L}$ for $T = 2L$, because we do not have any version of Levinson theorem in which the appearance of non-trivial Jordan box for $S_\infty = C_\lambda$ can be accepted. Nevertheless, as we shall see, the above asymptotic results are strong enough to prove important spectral results for J.

(b) For cases (i) and (ii) and $\alpha \in (0, 1)$ formula (2.12) does not give any strong estimate for $|u_m(k)|$, because (2.13) and $\operatorname{re}\eta_{m\infty} = 0$ follow that

$$|1 + n^{-\alpha}\eta_m(n)| = 1 + n^{-\alpha}\varepsilon(n) \quad \text{with } \varepsilon(n) \to 0,$$

and no further information on $\varepsilon(n)$ can be found on the base of the first part of the theorem. Consequently no much stronger estimates than

$$|u_m(nT+s)| = o(\exp(\varepsilon_0 n^{1-\alpha}))$$

for any $\varepsilon_0 > 0$ can be obtained this way. Much better estimate follows from (2.14). Namely,

$$|u_m(nT+s)| = O\left(\frac{1}{\sqrt{b(nT-1)}}\right) = O(n^{-\alpha/2}). \tag{2.16}$$

In case (iii) formula (2.12) contains a key information for us. Since $\eta_{1\infty} < 0$ for $\alpha \in (0;1)$ and $\eta_{1\infty} < -\frac{1}{2}$ for $\alpha = 1$, there exists $g > \frac{1}{2}$ such that $|1 + n^{-\alpha}\eta_1(n)| < |1 - gn^{-1}|$ for large n, and thus

$$|u_1(nT+s)| = O(n^{-g}) \tag{2.17}$$

(for $\alpha < 1$ better estimates can be found). Therefore, in particular, $u_1 \in l_1^2$.

Let us concentrate now on the spectral applications of the asymptotic results for J. The following result is obtained by the use of subordination theory of Khan and Pearson [14].

Theorem 2.2. *Let J be determined by* (2.7). *If $T = 2L+1$, then J is absolutely continuous and $\sigma_{ac}(J) = \mathbb{R}$. If $T = 2L$ and ρ is given by* (2.10), *then J is absolutely continuous in $\mathbb{R}\setminus[-\frac{|\rho|}{L}; \frac{|\rho|}{L}]$, $\sigma_{ac}(J) = \mathbb{R}\setminus(-\frac{|\rho|}{L}; \frac{|\rho|}{L})$, and J is purely point in $[-\frac{|\rho|}{L}; \frac{|\rho|}{L}]$.*

Proof. Let us study first the subordinated solutions of (2.1) (see [14]). Define (we use "s.s." for subordinated solution)

$$S_{ac} = \{\lambda \in \mathbb{R}: \text{ there is no s.s. of (2.1) for } \lambda\},$$

$$S_{sing} = \{\lambda \in \mathbb{R}: \text{ the s.s. } u \text{ of (2.1) for } \lambda \text{ exists, and } (\mathcal{J}u)(1) = \lambda u(1)\}.$$

For cases (i) and (ii) of Theorem 2.1 the subordinated solution does not exist, by the Janas and Naboko generalization of Behncke–Stolz lemma (see e.g. [10, Lemma 1.5]) and by (2.16). Thus, for $T = 2L+1$ $S_{ac} = \mathbb{R}$, and for $T = 2L$ $S_{ac} \supset \mathbb{R}\setminus[-\frac{|\rho|}{L}; \frac{|\rho|}{L}]$. Let $T = 2L$. By (2.17), for $|\lambda| < \frac{|\rho|}{L}$ we have $u_1 \in l_1^2$, which implies that u_1 is the subordinated solution. Thus $S_{ac} \subset \mathbb{R}\setminus(-\frac{|\rho|}{L}; \frac{|\rho|}{L})$, $S_{sing} \subset [-\frac{|\rho|}{L}; \frac{|\rho|}{L}]$. Moreover, if $\lambda \in S_{sing} \cap (-\frac{|\rho|}{L}; \frac{|\rho|}{L})$ then $\lambda \in \sigma_{pp}(J)$ and u_1 is the eigenvector for J and λ. Hence S_{sing} is a countable set. Therefore, using Khan and Pearson Theorem from [14] (see also [16, Theorem 1.1]) we obtain:

- for $T = 2L+1$ $\mathcal{H}_{ac}(J) = l_1^2$, $\sigma_{ac}(J) = \mathbb{R}$,
- for $T = 2L$ $\mathcal{H}_{\mathbb{R}\setminus[-\frac{|\rho|}{L};\frac{|\rho|}{L}]}(J) = \mathcal{H}_{ac}(J)$ (since we have "$\subset$" and $\mathcal{H}_{\left[-\frac{|\rho|}{L};\frac{|\rho|}{L}\right]}(J) \cap \mathcal{H}_{ac}(J) = \mathcal{H}_{\left(-\frac{|\rho|}{L};\frac{|\rho|}{L}\right)}(J) \cap \mathcal{H}_{ac}(J) = \{0\}$, because $\mathcal{H}_{\{\pm\frac{|\rho|}{L}\}}(J)$ is

an eigenspace, or the zero space), $\sigma_{ac}(J) = \mathbb{R}\setminus(-\frac{|\rho|}{L}; \frac{|\rho|}{L})$, and $\mathscr{H}_{[-\frac{|\rho|}{L};\frac{|\rho|}{L}]}(J) = \mathscr{H}_{pp}(J)$ (since $\mathscr{H}_{sc}(J) = \{0\}$ by countability of S_{sing}). $\square$

The above result partially generalizes the result of [16] on "the spectral gap" for Jacobi matrix with double weights, being the special case of J studied here (for $T = 2$, $\alpha = 1$, and an appropriately chosen $c(1)$ and c(2)); see also [4]. It remains an open problem if the spectrum of J in the gap $(-\frac{|\rho|}{L}; \frac{|\rho|}{L})$ is discrete (or even finite), as it is, e.g. in the double weights case.

Note that a proof of the assertion on the absolute continuity in the above theorem can be also obtained by the use of the so-called H-class of sequences of 2×2 matrices—see [16].

2.2. A perturbation of rapidly increasing weights

The next class of Jacobi matrices which can be studied with the help of discrete Levinson theorems is given by faster increasing weights. Namely let

$$b(n) = n^{\alpha}\left(1 + \frac{c(n)}{n}\right), \quad n \geqslant 1 \tag{2.18}$$

where $\alpha > 1$ and $c = \{c(n)\}_{n\in\mathbb{Z}}$ is a T-periodic real sequence, with an even period $T = 2L$, and $(1 + \frac{c(n)}{n}) \neq 0$ for $n \geqslant 1$. Since such weight sequence do not satisfy the Carleman condition, we shall assume also self-adjointness of J. To assure that this is possible (note that J is not self-adjoint e.g. for $c(n) \equiv 0$), we prove the following lemma being a generalization for arbitrary L of an example found by Kostyuchenko and Mirzoev [15] (where $L = 1$, but with the weights of more general form). Define (cf. also (2.10))

$$\rho = \sum_{j=1}^{2L} (-1)^j c(j).$$

Lemma 2.3. *J is self-adjoint provided $|\rho| \geqslant L(\alpha - 1)$.*

Proof. By Berezanskii [2, Chapter V, Lemma 1.5], it is enough to check that there exists a solution to (2.1) which is not in l_1^2 for some $\lambda \in \mathbb{C}$. Taking $\lambda = 0$ and computing the two solutions with boundary conditions $(u(1), u(2)) = (0, 1)$ or $(1, 0)$, we see that this is true when one of the following conditions holds:

$$\text{(i)} \quad \sum_{n=1}^{+\infty}\left(\prod_{j=1}^{n} \frac{b(2j-1)}{b(2j)}\right)^2 = +\infty; \quad \text{(ii)} \quad \sum_{n=1}^{+\infty}\left(\prod_{j=1}^{n} \frac{b(2j)}{b(2j+1)}\right)^2 = +\infty.$$

Using the Gauss test (for the series being the sums of the terms of the above series for "$n = Lk$") we can compute that the condition $\rho \geqslant L(\alpha - 1)$ gives (ii) and $-\rho \geqslant L(\alpha - 1)$ gives (i). $\square$

Similarly as in the previous subsection, we want to compute now an asymptotic formula for the sequence of grouped transfer matrices $A(n)$ defined by (2.5). In this case these computations are much simpler, since we have $\alpha > 1$ (and thus $\{1/n^\alpha\} \in l^1$). Indeed, for any $j = 0, \dots, T-1$, $n \geqslant 1$ we can easily obtain

$$B(nT+j) = E + n^{-1}\begin{pmatrix} 0 & 0 \\ f_j & 0 \end{pmatrix} + R_j(n),$$

where $f_j = T^{-1}(\alpha + c(j) - c(j-1))$ and $\{R_j(n)\}_{n\geqslant 1} \in l^1$. Thus for $n \geqslant 1$

$$A(n) = (-1)^L + n^{-1}S_\infty + R(n),$$

where $\{R(n)\}_{n\geqslant 1} \in l^1$ and (by the calculations similar to these used in the proof of Proposition 2.1)

$$S_\infty = (-1)^{L-1}\begin{pmatrix} \frac{\alpha}{2} + \frac{\rho}{T} & 0 \\ 0 & \frac{\alpha}{2} - \frac{\rho}{T} \end{pmatrix}. \tag{2.19}$$

Thus S_∞ does not depend on λ and $\operatorname{discr} S_\infty > 0$, if $\rho \neq 0$.

We can now formulate the theorem on asymptotic behaviour of a generalized eigensolution and of spectral properties for J.

Theorem 2.3. *If $\rho \neq 0$, then the generalized eigenequation* (2.1) *for J determined by* (2.18) *and for $\lambda \in \mathbb{R}$ has a non-zero solution u of the form*

$$u(nT+s) = \left(\prod_{j=1}^{n-1}\left(1 - j^{-1}\left(\frac{\alpha}{2} + \frac{|\rho|}{T}\right)\right)\right)\beta(nT+s), \tag{2.20}$$

$n \geqslant 0$, $s = 1, \dots, T$, *where* $\lim_{k\to+\infty} \beta(k) - \tilde\beta(k) = 0$, *with* $\tilde\beta$ *being the* 4*-periodic sequence defined by*:

$$(\tilde\beta(0), \tilde\beta(1), \tilde\beta(2), \tilde\beta(3)) = \begin{cases} (0, -1, 0, 1) & \textit{for } \rho > 0, \\ (1, 0, -1, 0) & \textit{for } \rho < 0. \end{cases} \tag{2.21}$$

If, moreover, J is self-adjoint, then J is a purely point operator.

Proof. Define $\mu_{1\infty} := (-1)^{L-1}(\frac{\alpha}{2} + \frac{|\rho|}{T})$ and $\mu_{2\infty} := (-1)^{L-1}(\frac{\alpha}{2} - \frac{|\rho|}{T})$, and observe that $\mu_{1\infty}, \mu_{2\infty}$ are the eigenvalues of S_∞ satisfying $(-1)^L(\mu_{2\infty} - \mu_{1\infty}) > 0$. Thus, by Theorem 1.7(b) and by (2.6) Eq. (2.1) has a non-zero solution u of the form (obtained after multiplying by $(-1)^L$ of the solution from Theorem 1.7)

$$u(nT+s) = \left(\prod_{j=1}^{n-1}\left(1 - j^{-1}\left(\frac{\alpha}{2} + \frac{|\rho|}{T}\right)\right)\right)(-1)^{nL}\left(\prod_{j=0}^{s-1} B(nT+j)v(n)\right)_2,$$

$n \geqslant 0$, $s = 1, \dots, T$, where $v(n) \to \begin{cases} e_1 & \text{for } \rho > 0, \\ e_2 & \text{for } \rho < 0. \end{cases}$ Since $B(n) \to E$, we have (2.20) with $\tilde\beta(nT+s) = (-1)^{nL}(E^s e_j)_2$, where $j = 1$ for $\rho > 0$ and $j = 2$ for $\rho < 0$. Thus using

$E^{nT+s} = E^{2nL+s} = (-1)^{nL}E^s$ we obtain $\tilde{\beta}(k) = (E^k e_j)_2$, which gives (2.21). Now, observe that by (2.20), $u \in l_1^2$, since $(\frac{\alpha}{2} + \frac{|\rho|}{T}) > \frac{1}{2}$. Therefore, using the same subordination theory arguments as in the proof of Theorem 2.2 we obtain the last assertion of our theorem. $\square$

2.3. A Jacobi matrix with divergent sequence of transfer matrices

The aim of this section is to present an example of Jacobi matrix, for which "the discrete Hartman–Winter theorem" (Theorem 1.3) can be used to obtain the asymptotics of solutions of the generalized eigenequation.

We follow here the notation and the definitions of the previous section. We only change the formula for the weights $b(n)$. Let $\{a(n)\}_{n\geqslant 1}$ be a real sequence satisfying $a(n) \neq 0$, $a(n) \to a_\infty \in \mathbb{R} \setminus \{-1, 0, 1\}$, let $\alpha \in (\frac{1}{2}, 1]$, and for $n \geqslant 1$ define

$$b(2n) = \prod_{j=1}^{n} \left(1 + \frac{\alpha}{j}\right), \quad b(2n-1) = b(2n)a(n). \tag{2.22}$$

We have

$$\prod_{j=1}^{n} \left(1 + \frac{\alpha}{j}\right) = c_\alpha n^\alpha + o(n^\alpha), \quad c_\alpha > 0, \tag{2.23}$$

which, by the Carleman criterion, gives the self-adjointness of J (see e.g. [2]). In the previous section the sequence of the transfer matrices was convergent to the limit E. Here $\{B(n)\}_{n\geqslant 2}$ is not convergent, but we have

$$B(2n) \to \begin{pmatrix} 0 & 1 \\ -a_\infty & 0 \end{pmatrix}, \quad B(2n+1) \to \begin{pmatrix} 0 & 1 \\ -a_\infty^{-1} & 0 \end{pmatrix}. \tag{2.24}$$

Proposition 2.2. *Let J be determined by* (2.22). *The generalized eigenequation* (2.1) *for J and $\lambda \in \mathbb{R}$ has two linearly independent solutions u_1, u_2 of the form*

$$u_1(2n+s) = \left(\prod_{j=1}^{n-1} a(j)\right) \beta_1(2n+s),$$

$$u_2(2n+s) = n^{-\alpha} \left(\prod_{j=1}^{n-1} a(j)\right)^{-1} \beta_2(2n+s)$$

for $n \geqslant 0$, $s = 1, 2$, *such that* $\lim_{k\to+\infty} \beta_m(k) - \tilde{\beta}_m(k) = 0$, $m = 1, 2$, *with* $\{\tilde{\beta}_m(k)\}_{k\geqslant 1} \in l(\mathbb{R})$ *being the* 4*-periodic sequence given by*

$$\tilde{\beta}_m(k) = \begin{cases} 0 & \text{for } k = 2n + m - 1, \\ (-1)^n & \text{for } k = 2n + m. \end{cases}$$

Moreover J is a purely point operator.

Proof. Define $A(n) := B(2n+1)B(2n)$ for $n \geqslant 1$ (i.e. $A(n)$ is given by (2.5) with $T=2$). We have $A(n) = \Lambda(n) + R_{\mathrm{AD}}(n) + R_{\mathrm{D}}(n)$ where

$$\Lambda(n) = \begin{pmatrix} -a(n) & 0 \\ 0 & -(a(n+1)(1+\frac{\alpha}{n+1}))^{-1} \end{pmatrix},$$

$$R_{\mathrm{AD}}(n) = \begin{pmatrix} 0 & \lambda(b(2n))^{-1} \\ -\lambda a(n)(b(2n+1))^{-1} & 0 \end{pmatrix},$$

$$R_{\mathrm{D}}(n) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^2(b(2n)b(2n+1))^{-1} \end{pmatrix},$$

and by (2.22) and (2.3), for (2.4) all the assumptions of Theorem 1.3 are satisfied. Thus, using (2.6) (which is valid for any weight sequence) with $T=2$ we obtain two linearly independent solutions of (2.1) of the form

$$u_1'(2n+s) = \left(\prod_{j=1}^{n-1} a(j)\right)(-1)^{n-1}\left(\prod_{j=0}^{s-1} B(2n+j)v_1(n)\right)_2,$$

$$u_2'(2n+s) = \left(\prod_{j=1}^{n-1} (a(j+1)\left(1+\frac{\alpha}{j+1}\right))^{-1}\right)(-1)^{n-1} \times \left(\prod_{j=0}^{s-1} B(2n+j)v_2(n)\right)_2$$

for $n \geqslant 0,\ s=1,2$, where $v_m(n) \to e_m$. By (2.23) and (2.24) we have

$$u_1'(2n+s) = \left(\prod_{j=1}^{n-1} a(j)\right)((-1)^{n-1}\delta_1(s) + \varphi_1(2n+s)),$$

$$u_2'(2n+s) = n^{-\alpha}\left(\prod_{j=1}^{n-1} a(j)\right)^{-1}((-1)^{n-1}c\delta_2(s) + \varphi_2(2n+s)),$$

where c is a non-zero constant, $\varphi_m(k) \to 0$ and

$$\delta_1(s) = \begin{cases} -a_\infty & \text{for } s=1, \\ 0 & \text{for } s=2, \end{cases} \qquad \delta_2(s) = \begin{cases} 0 & \text{for } s=1, \\ -a_\infty^{-1} & \text{for } s=2. \end{cases}$$

Observe, that for $k=2n+s$, with $s=1,2,\ n \geqslant 0$ we have $(-1)^{n-1}\delta_m(s) = c_m'\tilde{\beta}_m(k)$ (for $\tilde{\beta}_m$ defined as in the assertion of the theorem), for some non-zero constants $c_m',\ m=1,2$. This proves "the asymptotic part" of the theorem. To prove that J is purely point we can proceed analogously to the proof of Theorem 2.2 for $T=2L$, since for any $\lambda \in \mathbb{R}$ one of the solutions u_1, u_2 is in l_1^2 (u_1 if $|a_\infty| < 1$, and u_2 if $|a_\infty| > 1$) (and hence $S_{\mathrm{ac}} = \phi$, and $\lambda \in S_{\mathrm{sing}}$ iff $\lambda \in \sigma_{\mathrm{pp}}(J)$). $\square$

Note, that Theorem 1.3 is the unique result of Section 1 which could be used directly to study Eq. (2.4) (for $A(n)$ defined in the above proof), since $\{R_{\mathrm{AD}}(n)\}_{n\geqslant 1} \notin l^1$ for $\lambda \neq 0$. We could also try to use Theorem 1.5 with $V(n) = \Lambda(n) + R_{\mathrm{AD}}(n)$, but the condition $\{a(n)\}_{n\geqslant 1} \in D^1$ is necessary then.

Note also that the proof of the spectral part of Proposition 2.2 can be also obtained by the use of some weaker estimates for the l^2 solution of the generalized eigenequation. For instance, Freiman's generalization of the Perron theorem for discrete systems (see [8]) can be used there.

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